Forschungserfahrungen im Bachelor-Studium

Linearizations of Inequalities for Mixed Discriminants (Projekt auf Englisch)

Mathematical Subject Classification:

52A39, 15B48, 15A15, 15A45, 52A20, 52A40

The general research framework for the project is the interplay between mixed discriminants of positive semidefinite matrices and mixed volumes of convex bodies (convex and compact sets) in $\mathbb{R}^n$. Mixed descriminants of positive semidefinite matrices $A, B \in \mathcal{M}(n, \mathbb{R})$ arise naturally when addressing $\det(A+B)$. In a parallel way, the mixed volumes of two convex bodies $K, L \subset \mathbb{R}^n$ are found when computing $\mathrm{vol}_n(K+L)$, where the sum here is the usual sum in $\mathbb{R}^n$.

As a consequence of a classical result for positive semidefinite matrices A, B, the inequality $\det(A+B) \geq \det(A) + \det(B)$ follows immediately. In the geometrical counterpart, it is also known that $\mathrm{vol}_n(K+L) \geq \mathrm{vol}_n(K) + \mathrm{vol}_n(L)$ for convex bodies K, L. Let now $\lambda \in [0,1]$. Then $\lambda A + (1-\lambda)B$ is a positive semidefinite matrix for any $\lambda \in [0,1]$, and $\lambda K + (1-\lambda)L$ is a convex body for any $\lambda \in [0,1]$. Although the inequality $\mathrm{vol}_n(\lambda + (1-\lambda)L) \geq \lambda \mathrm{vol}_n(K) + (1-\lambda)\mathrm{vol}_n(L)$ is known not to be true in general for all convex bodies K, L, and all $\lambda \in [0,1]$, there are well-defined conditions on K, L which allow to have this inequality for all $\lambda \in [0,1]$, as equal (volume of) projections onto hyperplanes.

If A, B are positive semidefinite matrices, it is natural to wonder whether the inequality $\det(\lambda A + (1-\lambda)B) \geq \lambda \det(A) + (1-\lambda)\det(B)$ is satisfied. It is enough to consider the positive semidefinite matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}$ to realize, that, as it happens in the geometric setting, further assumptions on the matrices need to be taken. Indeed, since positive semidefinite matrices uniquely define ellipsoids, it is geometrically clear, that those conditions are necessary, from the connection of the determinant of a positive semidefinite matrix and the volume of the ellipsoid defined by it.

The main aim of this project is to study appropriate conditions on the positive semidefinite matrices A, B, or related matrices, such that for all $\lambda \in [0,1]$ the inequality $\det(\lambda A + (1-\lambda)B) \geq \lambda \det(A) + (1-\lambda)\det(B)$ holds true.

References

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